Bit geometrically uniform encoders: a systematic approach to the design of serially concatenated TCM

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Abstract — **We introduce a class of encoders, called *bit geometrically uniform* (BGU) encoders, for which the bit error probability does not depend on the transmitted sequence. Strong connections between the symmetry groups of geometrically uniform signal constellations and those of the binary Hamming spaces are involved. Both uncoded modulation and infinitely long trellis codes are considered, and connections with the code linearity discussed. The theory of BGU encoders proves very useful for the analysis and design of codes aimed at minimizing the *bit*, rather than *symbol*, or *sequence* error probability. We apply the theory to the design of good serially concatenated trellis-coded modulation schemes.**

serial concatenation

I. Introduction

A geometrically uniform (GU) constellation [9] is a set of signals for which the Uniform Error Property (UEP) holds: the Voronoi (decision) regions of the signals are all congruent and the *symbol* error probability is the same for each transmitted signal. Most of the constellations used in practical applications are either GU (PSK), or approximately GU if we neglect boundary effects (PAM, QAM).

The notion of geometrical uniformity has been extended in [9] to codes of finite or infinite length over GU constellations. Also in this case, the Voronoi regions of GU code sequences are congruent and the symbol sequence error probability (often called *error event* probability) can be computed from any sequence.

For most transmission systems, the *bit error probability* is a very important performance measure, which involves the reliability of the binary information sequences that must be delivered to their final destination. While a *code* can be defined as the set of signal sequences that are transmitted over the channel, the *encoder* is the machine that associates the binary information sequences to the code sequences.

In classical coding theory, encoders are often not considered, or somehow confused with the code, and the attention in designing a good code is focused on the (Hamming or Euclidean) distance properties of the code. In some applications, however, a clear separation between codes and encoders is essential (e.g. the concept of rotational invariance [11], [3]), or different encoders generating the same code lead to significantly different performance (e.g. constituent codes of turbo codes [8], [5]).

Turbo codes [8], and other forms of code concatenations with interleavers [7] are newly discovered codes yielding astonishing performance in terms of bit error probability. Tools for their analysis and design are now well established [5, 4], and rely heavily on their being linear codes possessing the uniform error property both at word and bit level.

Extension of turbo codes to bandwidth efficient trellis-coded modulations (TCM) has been already tempted. The proposed schemes, based on handcrafted constructions [6, 1, 10], yield very good performnce, but lack of a systematic approach. The main reason for the difficulty of extending the design technique valid for binary turbo codes to TCM schemes is that they, although GU, do not possess the uniform *bit* error property, so that the bit error probability depends on the transmitted sequence.

The goal of this paper is to extend the concept of UEP to the binary information sequences, and find the conditions under which also the bit error probability can be computed by considering any transmitted sequence, for example the all-zero sequence. While UEP applies to the code, this property, which will be called *uniform bit error property* (UBEP) in the following, applies to the encoder. We introduce the notion of *bit geometrically uniform* (BGU) encoders. prove that a BGU encoder satisfies the UBEP, and derive necessary and sufficient conditions under which the trellis section corresponds to a BGU encoder.

UBEP proves very useful in all applications where the input/output relationships between binary information sequences and signal sequences are important, because it simplifies the performance evaluation and the design of codes. In this paper, we derive some conditions under which an encoder satisfies UBEP, and apply this concept to the perfomance evaluation and design of good serially concatenated TCM codes [1], a technique that provides very good performance at high spectral efficiency via iterative decoding strategies. As for all concatenated codes with interleavers, whose aim is the minimization of the bit error probability obtained acting on the multiplicities of near neighbours, the input/output relationship of constituent encoders has a strong impact on performance, and UBEP properties highly simplify the search for good codes.

II. Serially concatenated trellis coded modulation

Serial concatenation of an outer binary convolutional encoder with an inner trellis encoder over a multidimensional Euclidean constellation through an interleaver, and a suitable iterative decoding algorithm were proposed in [1] with some examples of codes with very good performance. A serially concatenated trellis coded modulation (SCTCM) consists of an outer binary convolutional encoder with rate $a/(a+1)$; the output bits, an interleaver and a TCM encoder defined over a $2L$-dimensional constellation with a spectral efficiency of $(a+1)/L$ bps/Hz (with ideal Nyquist pulse shaping). The spectral efficiency of the overall SCTCM is then a/L bps/Hz.

In [1], extending the approach introduced in [5] for binary encoders, the design approach was based on the assumption of a large uniform interleaver, and led to the requirement of maximizing the *effective free Euclidean distance* of the inner TCM code $\mathcal{C}$, defined as the minimum distance between code sequences generated by information sequences that differ only by two bits:

$$d_{\mathrm{f,eff}} = \min_{\mathbf{c}_1,\mathbf{c}_2} d_E(\mathbf{c}_1,\mathbf{c}_2) \quad \forall \mathbf{c}_1,\mathbf{c}_2 \in C \; : \; d_H(E(\mathbf{c}_1),E(\mathbf{c}_2)) = 2 \tag{1}$$

where d_E means Euclidean distance. The inner TCM encoder must also be *recursive*, *i.e.*, no finite-weight code sequence must be generated by an input sequence of weight one.

According to the definition, the computation of $d_{\mathrm{f,eff}}$ requires in general testing of all possible pairs $(\mathbf{c}_1,\mathbf{c}_2)$. However, if the inner TCM encoder is BGU, we can choose as input sequence the all-zero sequence and compute $d_{\mathrm{f,eff}}$ as

$$d_{\mathrm{f,eff}} = \min_{\mathbf{c}} w_E(\mathbf{c}) \quad \text{for all} \quad \mathbf{c} \in C \quad \text{with} \quad w_H(E(\mathbf{c})) = 2 \tag{2}$$

where by $w_E(\mathbf{c})$ we denote the Euclidean distance between the code sequence $\mathbf{c}$ and the all-zero sequence. The great simplification involved in passing from (1) to (2) is apparent. Moreover, since the search for good SCTCM codes also involved maximization of minimum Euclidean distance for pairs of input sequences with Hamming distances larger than 2, typically up to 6, the reduction of the computational burden involved in the search becomes dramatic.

Moreover, analytical upper bounds to the ML bit error probability performance for BGU SCTCM codes can be obtained as a straightforward extension of the technique developed in [5].

Owing to these simplifications, we have performed a complete search, starting from the best 2-state 2.5 bps/Hz GU TCM codes over 2×8-PSK constellation of [2] (the same code used in [1]), and constructed all possible BGU recursive encoders for this code. For each encoder we have computed $d_{\mathrm{f,eff}}$ (this computation is highly simplified for encoders possessing the BGU properties because definition (2) can be applied), and selected the best ones according to their input Hamming-distance-output Euclidean distance. More precisely, we have selected the ones for which the pairs (d_i, N_i) are optimized (d_i maximized and N_i minimized), from $i = 2$ up to $i = 10$.

The analytical upper bounds to the bit error probability, evaluated through an extension of the technique described in [5] for four SCTCMs of spectral efficiency 2 bps/Hz employing as outer code a binary convolutional code of rate 4/5, 2 states, a uniform interleaver [5] with length $N = 100$, or $N = 1000$, and as inner encoders four different TCM encoders of spectral efficiency 2.5 bps/Hz over a 2×8-PSK are shown in Fig. 1. Curve A referes to the best encoder found through the search, that sequentially optimizes (d_i, N_i), for $i = 2, ..., 10$, curve B represents the performance of the code of [1]; curve C pertains to a second encoder found that sequentially maximizes d_i, for $i = 2, ..., 10$, without considering the multiplicities; curve D referes to a third encoder found that maximizes d_4 (this is relevant because the outer code has minimum distance 3).

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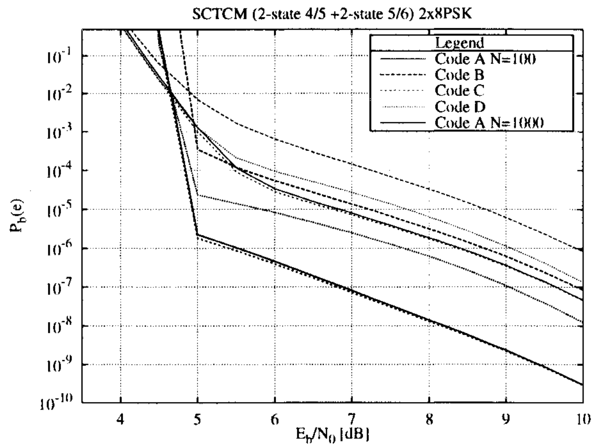

Figure 1: Upper bounds to the bit error probability for four SCTCM codes of spectral efficiency 2 bps/Hz employing a uniform interleaver with length $N = 100$, or $N = 1000$.

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